

Comment on “New conditions for a total neutrino conversion in a medium”

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Abstract

We show that the conditions for total neutrino conversion found in [1] are equivalent to the conditions of maximal depth (parametric resonance) and $(\pi/2 + \pi k)$ - phase of parametric oscillations. Therefore the effects considered in [1] are a particular case of the parametric resonance in neutrino oscillations. The existence of strong enhancement peaks in transition probability P rather than the condition $P = 1$ is of physical relevance. We comment on possible realizations and implications of the parametric enhancement of neutrino oscillations.

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1. In the present Comment we show that the conditions for total neutrino conversion studied by Chizhov and Petcov [1] are just the conditions of the parametric resonance of neutrino oscillations supplemented by the requirement that the parametric enhancement be complete. Therefore the “new effect of total neutrino conversion” [1] is nothing but a particular case of the parametric enhancement of neutrino oscillations, suggested in [2, 3] and widely discussed in the literature [4, 5, 6, 7, 8].

The parametric resonance occurs when the oscillation frequency changes in a certain correlation with the frequency itself and with the amplitude of the oscillations, leading to specific phase relationships. A classical example is a pendulum with vertically oscillating suspension point [9]. This situation can, in particular, be realized for oscillating neutrinos crossing layers of medium of different densities [2, 3]. Indeed, the oscillation parameters depend on matter density, and crossing the layers of different density means changing frequency and amplitude of neutrino oscillations.

2. The propagation of neutrinos through medium with periodic density modulations leads to *parametric oscillations* [2, 3], see figs. 1 and 2. Let us consider neutrino propagation in a medium with the periodic “castle wall” density profile: a system of alternating layers of matter with constant densities N_1 and N_2 and widths L_1 and L_2 . Let $\theta_{1,2}$ be the mixing angles in matter at densities N_1 and N_2 . We denote by $2\phi_i$ ($i = 1, 2$) the oscillation phase acquired by neutrinos in the layer of density N_i and width L_i . We will use the notation $s_i \equiv \sin \phi_i$, $c_i \equiv \cos \phi_i$. The evolution matrix over one period of density modulation $L = L_1 + L_2$ is [6]

$$U_2 = Y - i\sigma\mathbf{X} = \exp[-i(\sigma\hat{\mathbf{X}})\Phi], \quad (1)$$

$$Y = c_1c_2 - \cos(2\theta_1 - 2\theta_2)s_1s_2, \quad \Phi = \arccos Y = \arcsin |\mathbf{X}|, \quad \hat{\mathbf{X}} = \mathbf{X}/|\mathbf{X}|. \quad (2)$$

The vector \mathbf{X} can be written in components as

$$\mathbf{X} = ((s_1c_2 \sin 2\theta_1 + s_2c_1 \sin 2\theta_2), -s_1s_2 \sin(2\theta_1 - 2\theta_2), -(s_1c_2 \cos 2\theta_1 + s_2c_1 \cos 2\theta_2)). \quad (3)$$

Notice that $Y^2 + \mathbf{X}^2 = 1$ as a consequence of unitarity of U_T .

From Eq. (1) one easily finds the transition probability after passing n periods [3, 6]

$$P(\nu_a \rightarrow \nu_b; r = nL) = \left(1 - \frac{X_3^2}{\mathbf{X}^2}\right) \sin^2 \Phi_p, \quad \Phi_p = n\Phi. \quad (4)$$

The transition probability after passing an odd number of alternating layers, which can be considered as n periods plus one additional layer of density N_1 (the corresponding distance $r = nL + L_1$), is also given by Eq. (4), the only difference being that the phase is now

$$\Phi_p = n\Phi + \varphi, \quad \varphi = \arcsin \left(s_1 \sin 2\theta_1 / \sqrt{1 - X_3^2/|\mathbf{X}|^2} \right). \quad (5)$$

Eqs. (4) and (5) give the transition probability at the borders of the layers. The pre-sine factor in (4) and Φ_p are the depth and the phase of the parametric oscillations. The phase Φ_p determines the length of the parametric oscillations (see figs. 1 and 2).

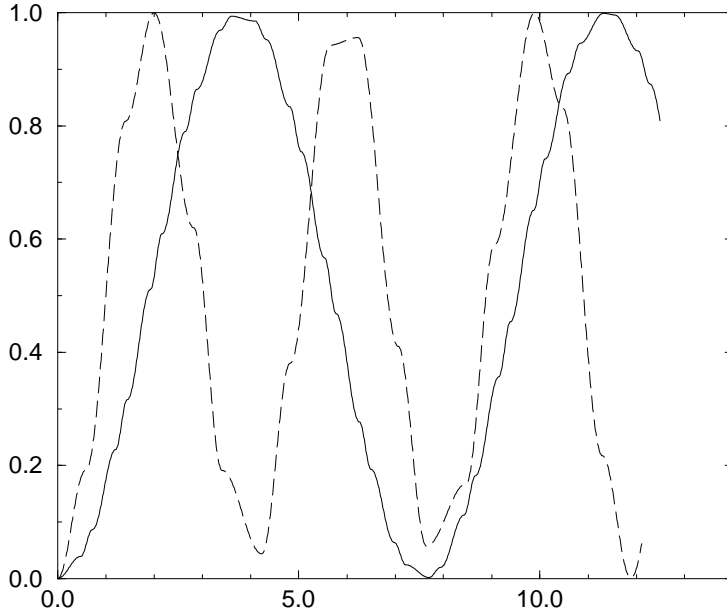


Figure 1: Parametric oscillations in a medium with “castle wall” density profile for the case $X_3 = 0$ (parametric resonance). Solid curve: transition probability for neutrino flavour oscillations as a function of the coordinate along the neutrino path for the case of total conversion over 5 periods of density modulation (10 layers). Dashed curve: the same for the case of total conversion over 3 layers. The kinks correspond to the borders of the layers of different densities. The curves were plotted for the realization (7) ($c_1 = c_2 = 0$) of the parametric resonance condition. Neutrino energy is between the MSW resonance energies corresponding to the densities N_1 and N_2 .

The parametric resonance occurs when the pre-sine factor in (4) becomes equal to unity, *i.e.* the depth of the parametric oscillations is maximal. The resonance condition is therefore (see Eq. (26) in [6])

$$X_3 \equiv -(s_1 c_2 \cos 2\theta_1 + s_2 c_1 \cos 2\theta_2) = 0. \quad (6)$$

The parametric resonance condition (6) can be realized in various ways¹. One well known realization [2, 3, 4, 5, 6] is $c_1 = c_2 = 0$, or

$$2\phi_1 = \pi + 2\pi k', \quad 2\phi_2 = \pi + 2\pi k'', \quad (7)$$

independently of the mixing angles². (This was reproduced as solution III in [1], see Eq. (18) there). If, however, c_1 and c_2 are non-zero, the cancellation between the two terms in

¹We do not consider the trivial cases of the MSW resonance for which $X_3 = 0$ because $\cos 2\theta_i = 0$ and $s_i = \pm 1$, $i = 1$ or 2 , or $\cos 2\theta_1 = \cos 2\theta_2 = 0$.

²It was renamed into “the oscillation length resonance” in [11].

(6) can occur, which implies certain correlation between the phases and mixing angles in the layers. (This covers solution IV in [1]). For an example, see fig. 2.

In general, the parametric resonance in neutrino oscillations does not require a periodic matter density profile (although the periodicity may make it easier to meet the resonance conditions), and can occur even in stochastic media [4].

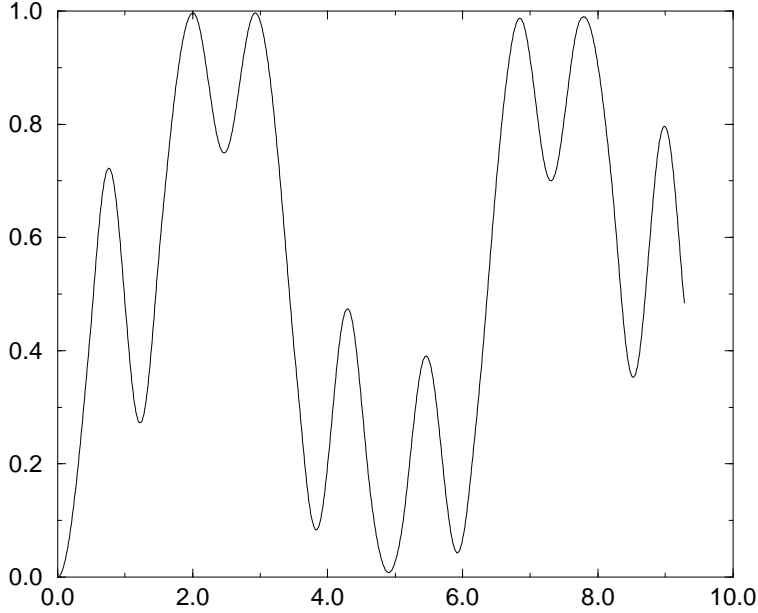


Figure 2: Same as in fig. 1 but for the case when the parametric resonance condition is realized through the cancellation of the two terms in Eq. (6). Total conversion is achieved over 3 layers. (Similar dependence of the transition probability can be obtained in the case $c_1 = c_2 = 0$ provided that the neutrino energy is above the MSW resonance energies corresponding to the densities N_1 and N_2).

3. As follows immediately from (4), the conditions for total neutrino conversion $P(\nu_a \rightarrow \nu_b) = 1$ are

$$X_3 = 0, \quad \Phi_p = \frac{\pi}{2} + 2\pi k \quad (8)$$

for evolution over any number of layers, including the two- and three-layer cases considered in [1]³. Thus *the maximal transition probability implies the fulfillment of the parametric resonance condition*.

³These cases correspond to $n = 1$ in Eqs. (4) and (5).

According to Eq. (4) for two layers the conditions (8) reduce to

$$X_3 = 0, \quad Y = 0. \quad (9)$$

It is easy to see that Eqs. (9) are equivalent to two conditions (22) in [1] (solution IV).

Notice that conditions (9) can be obtained directly from (1): the survival probability $P(\nu_a \rightarrow \nu_a) = Y^2 + X_3^2$ and therefore the condition of total neutrino conversion gives $Y^2 + X_3^2 = 0$.

Consider now the three-layer case (it gives a good approximation for the case of neutrinos crossing the earth, where the layers correspond to the mantle, core and then again mantle). Using expression for the phase (5), one can write the conditions of total transition (8) as $X_3 = 0$, $Y = \pm s_1 \sin 2\theta_1$, or equivalently as

$$X_3 = 0, \quad 2c_1 Y - c_2 = 0. \quad (10)$$

Conditions (10) can also be obtained directly from the evolution matrix which in this case is [6]

$$U_3 = Z - i\sigma \mathbf{W}. \quad (11)$$

Here

$$Z = 2c_1 Y - c_2, \quad (12)$$

Y has been defined in (2), and the vector \mathbf{W} can be written in components as

$$\mathbf{W} = (2s_1 Y \sin 2\theta_1 + s_2 \sin 2\theta_2, \quad 0, \quad -(2s_1 Y \cos 2\theta_1 + s_2 \cos 2\theta_2)). \quad (13)$$

The neutrino flavour transition probability in this case is $P(\nu_a \rightarrow \nu_b) = W_1^2$. The total neutrino conversion corresponds to zero survival probability: $Z^2 + W_3^2 = 0$, or

$$Z = 0, \quad W_3 = 0. \quad (14)$$

There are two possible realizations of these conditions, depending on the value of c_1 . If $c_1 = 0$ then from (12) and (14) it follows that c_2 must vanish, too. So, we arrive at the realization (7) of the parametric resonance condition (6). The second condition in (14) is then the one for the “totality” of transition. It can be written as $\cos(2\theta_1 - 2\theta_2) = \cos 2\theta_2 / 2 \cos 2\theta_1$, which is equivalent to the requirement that the transition probability [5]

$$P = \sin^2(2\theta_2 - 4\theta_1) \quad (15)$$

takes the value 1.

If $c_1 \neq 0$ then the first equality in (14) implies $Y = c_2 / 2c_1$. Inserting this into the expression for W_3 in (13) one obtains $W_3 = X_3 / c_1$. The condition $W_3 = 0$ thus means $X_3 = 0$. Therefore in this case, too, total neutrino conversion implies parametric resonance. Conditions (10) are equivalent at $c_1 \neq 0$ to the conditions of the total neutrino conversion in Eq. (26) of [1].

Similarly, one can analyze the case of $\nu_2 \rightarrow \nu_e$ transitions which is relevant for oscillations of solar and supernova neutrinos in the earth. In particular, it is easy to show that the parametric resonance condition for the probability P_{2e} of $\nu_2 \rightarrow \nu_e$ oscillations is

$$X'_3 \equiv X_3 \cos \theta_0 - X_1 \sin \theta_0 = 0. \quad (16)$$

The conditions of total $\nu_2 \rightarrow \nu_e$ conversion found in [1] imply equality (16).

4. The existence of strong enhancement peaks in transition probability P rather than the condition $P = 1$ is of physical relevance. For sufficiently large vacuum mixing angles, the transition probability has a series of peaks of comparable height and the total conversion peak is just one of them. In fact, peaks with $P_{max} < 1$ can contribute to observable effects even more than the ones with $P_{max} = 1$. For some applications, *e.g.*, for oscillations of solar neutrinos in the earth, even partial (or relative) enhancement can be important.

Let us comment on various realizations of the parametric enhancement of neutrino oscillations. Large oscillation effects can be due to large mixing in matter and therefore to large-amplitude oscillations, or due to specific properties of the density profile. In general, both mechanisms are present. Depending on neutrino parameters, either of the mechanisms can dominate, or they can give comparable contributions to the observable effects.

(i) The most interesting case is the one when neutrino mixing in matter of both densities N_1 and N_2 is small: $\sin^2 2\theta_1, \sin^2 2\theta_2 \ll 1$, and a strong enhancement of transition probability is due to the specific shape of the matter density distribution. Let us consider the three layer case (neutrino oscillations in the earth) with densities $N_1 - N_2 - N_1$ ($N_1 < N_2$) and concentrate on peaks of the transition probability with $P_{max} < 1$ relevant for solar neutrinos. Suppose that the neutrino energy is between the MSW resonance energies corresponding to the densities N_1 and N_2 which means that $2\theta_1 < \pi/2$ and $2\theta_2 > \pi/2$. In this case $\sin^2 2\theta_{1,2} \ll 1$ implies that $2\theta_1$ is small and $2\theta_2$ is close to π . If $4\theta_1 + (\pi - 2\theta_2) < \pi/2$ ⁴ the maximal enhancement of the transition probability takes place for the values of the oscillation phases $2\phi_i = \pi + 2\pi k_i$, *i.e.* for the realization (7) of the parametric resonance condition (6). In this case the transition probability given in (15) can be significantly larger than that in one layer with a matter of constant density with largest of the two $\sin^2 2\theta_i$ [11, 6].

If neutrino energy is above the MSW resonance energies, which means $2\theta_1, 2\theta_2 > \pi/2$, the smallness of $\sin^2 2\theta_{1,2}$ (even for large or maximal vacuum mixing) is due to the matter suppression effects. Again for $2(\pi - 2\theta_1) - (\pi - 2\theta_2) < \pi/2$ the maximal enhancement of the transition probability corresponds to the realization (7) of the parametric resonance with probability given in (15) [5].

Notice that for neutrinos traversing the earth the phases $2\phi_i$ are not arbitrary and the condition $2\phi_1, 2\phi_2 = (\text{odd integer}) \times \pi$ can be satisfied only approximately. For $2\phi_1, 2\phi_2 \neq (\text{odd integer}) \times \pi$ the transition probability is smaller than (15). In this case, for $\sin^2 2\theta_0 <$

⁴This condition is equivalent to $\cos(2\theta_2 - 4\theta_1) < 0$ [11, 6].

0.03, the maximum of P is achieved for relatively small but non-vanishing values of X_3 , which corresponds to the parametric oscillations with non-maximal depth.

(ii) For vacuum mixing close to the maximal one, $\sin^2 2\theta_0 \gtrsim 0.9$, the MSW resonances in the core and mantle are very wide and therefore the mixing angles in medium in the resonance energy interval are also large: $\sin^2 2\theta_1 \sim \sin^2 2\theta_2 \sim 0.9 - 1$. The change of the mixing angle in passing from the mantle to the core or vice versa is small and one can consider the earth matter as a single layer with a density close to the MSW resonance one. The effect of the matter density profile on the transition probability is small, and what matters is the total oscillation phase acquired when neutrinos traverse the earth. The complete conversion requires this phase to be an odd integer of π . In particular, for three layers this implies $2(2\phi_1 + \phi_2) = \pi(2k + 1)$. Indeed, large $\sin^2 2\theta_0$ solutions found in [10] satisfy this equality with a high precision.

(iii) There are several peaks of transition probabilities with $P(\nu_a \rightarrow \nu_b) = 1$ which correspond to intermediate values of the vacuum mixing angle, $\sin^2 2\theta_0 \simeq 0.15 - 0.6$ [10]. These peaks are due to an interplay of the effects of large-amplitude oscillations and specific matter density profile. However, none of the known neutrino anomalies can be explained through neutrino oscillations with mixing angles in this range. The oscillation solutions of the solar neutrino problem require the vacuum mixing angle to be either very small or close to the maximal one; the dominant mode of the atmospheric neutrino oscillations requires maximal or almost maximal mixing. The mixing angle θ_{13} governing the subdominant $\nu_e \leftrightarrow \nu_\mu$ and $\nu_e \leftrightarrow \nu_\tau$ oscillations of atmospheric neutrinos is severely restricted by the CHOOZ experiment [12] and the solar and atmospheric neutrino observations. Nevertheless, the solution with $\sin^2 2\theta_0 \simeq 0.15$ [10], though on the verge of being ruled out by CHOOZ for the range of Δm^2 allowed by the Super-Kamiokande atmospheric neutrino data, is at present not excluded. It can lead to a significant up-down asymmetry of the e-like events in the Super-Kamiokande atmospheric neutrino data (see fig. 7 in [7]).

We have shown that the effects discussed in [1] are those of the *parametric enhancement* of neutrino oscillations, contrary to the claim of the authors that they have found completely new effects which have nothing to do with the parametric resonance. Written in the form (9) or (10) the conditions for total neutrino conversion have a clear physical meaning of the conditions of the parametric resonance and $(\pi/2 + \pi k)$ phase of the parametric oscillations.

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